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Vibration analysis of twisted Timoshenko beams with internal Kelvin-Voigt damping

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Abstract

Bending-bending vibration equations of a twisted beam with damping of Kelvin-Voigt type are established using the Timoshenko beam theory and applying Hamilton's principle. The equations of motion of the twisted beam are derived in the twist coordinate frame. Then, a finite element method is used to reduce the partial differential equations of motion into linear second-order ordinary differential equations. A quadratic eigenvalue problem of a damped system is formulated to study the effects of the twist angle, internal damping and restraint types on the eigenfrequencies of the damped twisted beams.

Keywords: vibration; twisted beam; damping; Hamilton's principle; twist angle

1. Introduction

Dynamic analyses of the twisted structures, such as turbine blades, propeller blades and fluted cutters, have attracted many researchers' attention for the past years. The reasons result from the important applications of these structures in aircraft industry and electronic packaging technology, and the complex behaviour and modelling required for them. The dynamic behaviours of the twisted structures have been analyzed based on Euler beam theory [1-3] or Timoshenko beam theory [4-8]. Because the inclusion of damping has a profound impact on vibration characteristics of beam structures, the effects of internal and/or external lumped or distributed damping have been extensively investigated [9-13]. Recently, the studies [14-17] on the vibration of elastic systems with Kelvin-Voigt damping are increasing because of the applications of smart materials or passive controllers in modern technologies. Despite the extensive studies on vibration of beams, the vibration behaviour of twisted Timoshenko beams with internal Kelvin-Voigt damping has not yet been analyzed as the author knows.

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In the present study, bending-bending vibration equations of a twisted beam with Kelvin-Voigt damping are derived using the Timoshenko beam theory and applying Hamilton's principle. To represent the energy dissipation mechanism of Kelvin-Voigt damping, the dissipation function for the twisted beam has been developed based on the model by Kocatürk and Şimşek [15]. The equations of motion of the twisted beam are established in the twist coordinate frame. Then, a finite element method is used to reduce the partial differential equations of motion into a set of linear second-order ordinary differential equations with damping terms. A quadratic eigenvalue problem of a damped system is formulated to investigate the free vibration characteristics of the damped twisted beam. The effects of the twist angle, internal damping and restraint types on the eigenfrequencies of the damped twisted beams are discussed.

2. Bending vibration analysis

Figure 1 shows the model of twisted beam with Kelvin-Voigt damping and coordinate frames used. Coordinate systems XYZ and $\xi\eta Z$ represent the inertial and twist coordinate frame, respectively. The frame $\xi\eta Z$ moves along the beam twist angle such that axes ξ and η are in the principal directions of the beam cross-section.

To derive the bending vibration equations of motion of the Timoshenko beam, Hamilton's principle is applied to the Lagrangian (L) of the beam system to yield

$$\delta\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} (\delta T - \delta V + \delta W_d) dt = 0 \tag{1}$$

where T, V and δW_d are the total kinetic energy, the potential energy and the virtual work due to the internal damping, respectively. They are expressed in the inertial coordinate system XYZ as follows.

$$T = \frac{1}{2}\int_0^L [m(\dot{u}_X^2 + \dot{u}_Y^2)^2 + J_{XX}\dot{\varphi}_X^2 + 2J_{XY}\dot{\varphi}_X\dot{\varphi}_Y + J_{YY}\dot{\varphi}_Y^2] dZ \tag{2}$$

$$V = \frac{1}{2}\int_0^L \{\kappa GA[(u'_X - \varphi_Y)^2 + (u'_Y - \varphi_X)^2] + EI_{XX}(\varphi'_X)^2 + 2EI_{XY}\varphi'_X\varphi'_Y + EI_{YY}(\varphi'_Y)^2\} dZ \tag{3}$$

$$\delta W_d = \frac{\partial R}{\partial \dot{u}_X}\delta u_X + \frac{\partial R}{\partial \dot{u}_Y}\delta u_Y + \frac{\partial R}{\partial \dot{\varphi}_X}\delta\varphi_X + \frac{\partial R}{\partial \dot{\varphi}_Y}\delta\varphi_Y \tag{4}$$

Here R is the dissipation function of the twisted beam at any time instant. Based on the model by Kocatürk and Şimşek [15], the following dissipation function R for the twisted beam with Kelvin-Voigt damping is proposed.

$$R = \frac{1}{2}\int_0^L \left\{ C_b I_{XX}(\dot{\varphi}'_X)^2 + 2C_b I_{XY}(\dot{\varphi}'_X)(\dot{\varphi}'_Y) + C_b I_{YY}(\dot{\varphi}'_Y)^2 + C_s\kappa A\left[(\dot{u}'_X - \dot{\varphi}_Y)^2 + (\dot{u}'_Y - \dot{\varphi}_X)^2\right]\right\} dZ \tag{5}$$

where $C_b = E\eta_b$ and $C_s = G\eta_s$ denote the coefficients of the internal damping of the beam in bending and shearing, respectively; η_b and η_s are the proportionality constants of the internal damping of the beam.

By introducing Eqs. (2)-(5) into Eq. (1) and utilizing the transformation relationship [8] between the inertial and twist coordinates, the system equations for a twisted Timoshenko beam in the twist frame throughout the domain and the associated boundary conditions (at $z = 0, L$) could be obtained in matrix form as follows:

$$\overline{\mathbf{M}}\ddot{\mathbf{d}} + \overline{\mathbf{C}}_1\dot{\mathbf{d}}'' + (\overline{\mathbf{C}}_2 + \overline{\mathbf{C}}_3)\dot{\mathbf{d}}' + \overline{\mathbf{C}}_4\dot{\mathbf{d}} + \overline{\mathbf{K}}_1\mathbf{d}'' + (\overline{\mathbf{K}}_2 + \overline{\mathbf{K}}_3)\mathbf{d}' + \overline{\mathbf{K}}_4\mathbf{d} = \mathbf{0} \tag{6}$$

$$(\overline{\mathbf{C}}_1\dot{\mathbf{d}}' + \overline{\mathbf{C}}_2\dot{\mathbf{d}} + \overline{\mathbf{K}}_1\mathbf{d}' + \overline{\mathbf{K}}_2\mathbf{d}) = \mathbf{0} \text{ or } \mathbf{d} = \mathbf{0} \tag{7}$$

where $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}_i$ and $\overline{\mathbf{K}}_i$ are coefficient matrices associated with the system characteristics of the twisted beam structure; $\mathbf{d} = [u_\xi, u_\eta, \varphi_\xi, \varphi_\eta]^T$ is the displacement matrix of the twisted beam in the twisted frame; u_ξ, u_η, φ_ξ and φ_η are transverse displacements and angles of rotation, respectively. The explicit forms of matrices $\overline{\mathbf{M}}$ and $\overline{\mathbf{K}}_i$ can be found in Ref. [8]. The matrix $\overline{\mathbf{C}}_i$ is obtained by replacing E and G in $\overline{\mathbf{K}}_i$ with C_b and C_s, respectively.

3. Eigenfrequency analysis

Following a similar finite element approach [8], the governing partial differential equations (6) and (7) are discretized into a set of ordinary differential equations by using the Mindlin-type linear beam element. By substituting the displacement function of the beam element, $\mathbf{d}^{(e)}(z) = \mathbf{N}(z)\mathbf{p}^{(e)}$, into the weak form of Eqs. (6) and (7), and applying Galerkin's criterion, the following resulting beam element equations are obtained.

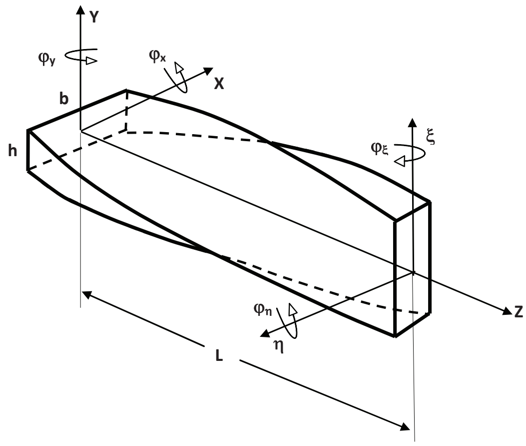


Fig. 1 Beam configuration and coordinate systems

$$\mathbf{M}^{(e)}\ddot{\mathbf{p}}^{(e)} + \mathbf{C}^{(e)}\dot{\mathbf{p}}^{(e)} + \mathbf{K}^{(e)}\mathbf{p}^{(e)} = \mathbf{0} \tag{8}$$

where

$$\mathbf{M}^{(e)} = \int_0^{L_e} \mathbf{N}^T\overline{\mathbf{M}}\mathbf{N}\,dz$$

$$\mathbf{C}^{(e)} = -\int_0^{L_e} \mathbf{B}^T\overline{\mathbf{C}}_1\mathbf{B}\,dz - \int_0^{L_e} \mathbf{B}^T\overline{\mathbf{C}}_2\mathbf{N}\,dz + \int_0^{L_e} \mathbf{N}^T\overline{\mathbf{C}}_3\mathbf{B}\,dz + \int_0^{L_e} \mathbf{N}^T\overline{\mathbf{C}}_4\mathbf{N}\,dz$$

$$\mathbf{K}^{(e)} = -\int_0^{L_e} \mathbf{B}^T\overline{\mathbf{K}}_1\mathbf{B}\,dz - \int_0^{L_e} \mathbf{B}^T\overline{\mathbf{K}}_2\mathbf{N}\,dz + \int_0^{L_e} \mathbf{N}^T\overline{\mathbf{K}}_3\mathbf{B}\,dz + \int_0^{L_e} \mathbf{N}^T\overline{\mathbf{K}}_4\mathbf{N}\,dz$$

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 & 0 & N_2 & 0 & 0 \\ 0 & 0 & N_1 & 0 & 0 & 0 & N_2 & 0 \\ 0 & 0 & 0 & N_1 & 0 & 0 & 0 & N_2 \end{bmatrix}, \quad \mathbf{B} = \mathbf{N}'$$

Here shape functions $N_1 = 1 - z/L_e$ and $N_2 = z/L_e$, where L_e is the length of the twisted beam element; $\mathbf{M}^{(e)}$, $\mathbf{C}^{(e)}$

and $\mathbf{K}^{(e)}$ are element mass matrix, element damping matrix and element stiffness matrix, respectively. In deriving the element stiffness of a Timoshenko beam by using C^0-order elements, the selective reduced integration method [18] is used to alleviate shear locking. The stiffness matrix due to bending effect is evaluated based on the normal quadrature rule, while the stiffness matrix due to shear effect is integrated with one-point quadrature scheme. In addition, the selective reduced integration method is also applied to derive the element damping matrix.

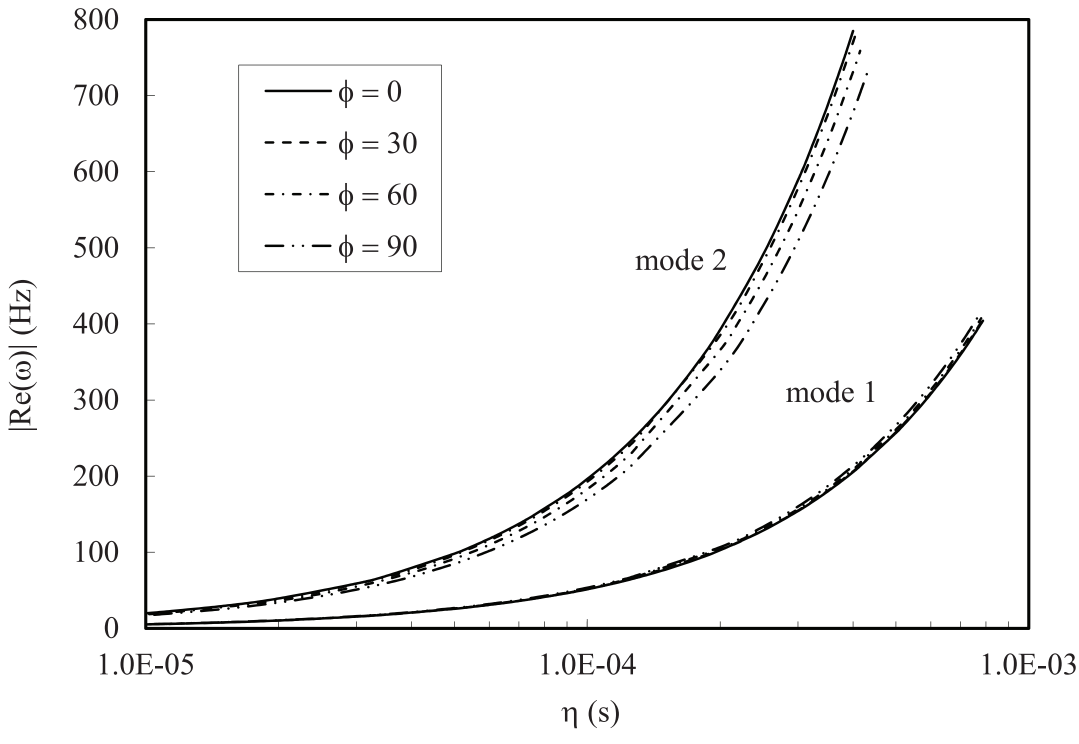
By directly assembling these element matrices and imposing the essential boundary conditions, the global system of equations can be obtained as

$$\mathbf{M}\ddot{\mathbf{p}} + \mathbf{C}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{0} \tag{9}$$

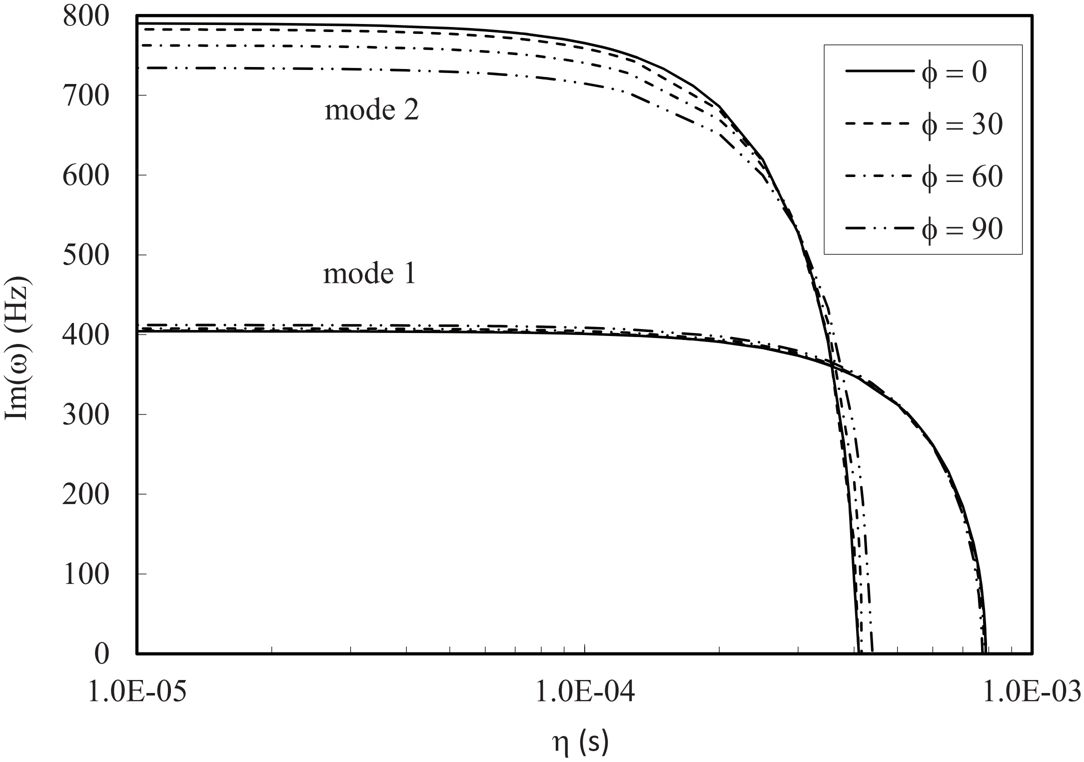
where **M**, **C** and **K** are the corresponding global matrices. With $\mathbf{p} = e^{\omega t}\mathbf{q}$, Eq. (9) is reduced to the following quadratic eigenvalue problem.

$$[\omega^2\mathbf{M} + \omega\mathbf{C} + \mathbf{K}]\mathbf{q} = \mathbf{0} \tag{10}$$

Here **q** is a constant vector and ω is the eigenfrequency associated with the damping system. Then, a standard linearization approach [19] is used to transform Eq. (10) into a 2n x 2n general eigenvalue problem as

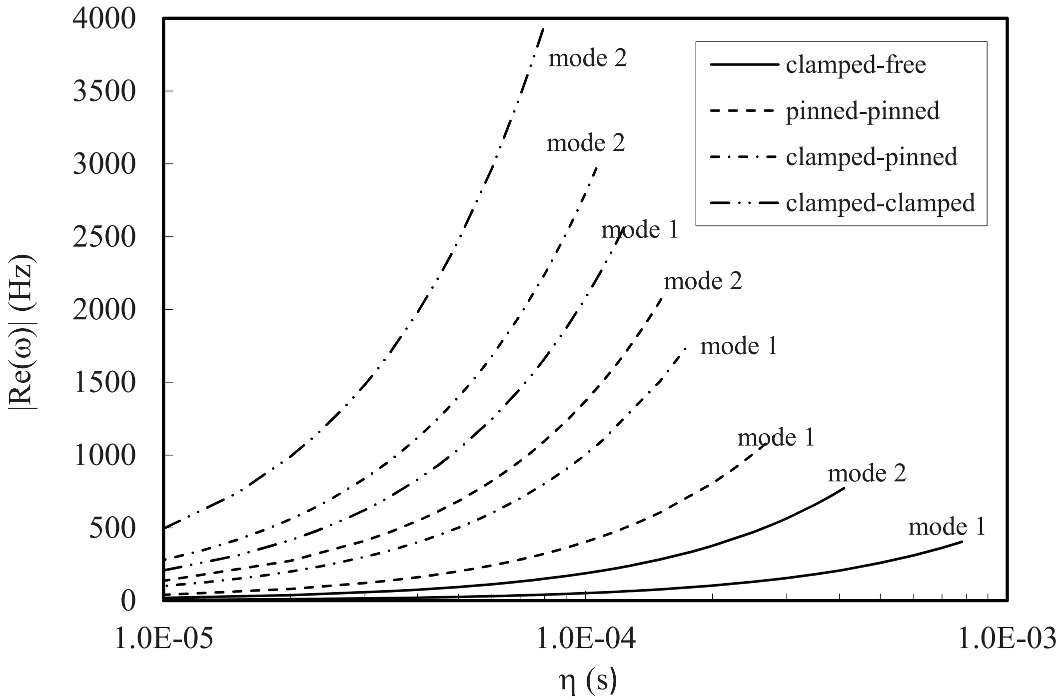


(a)

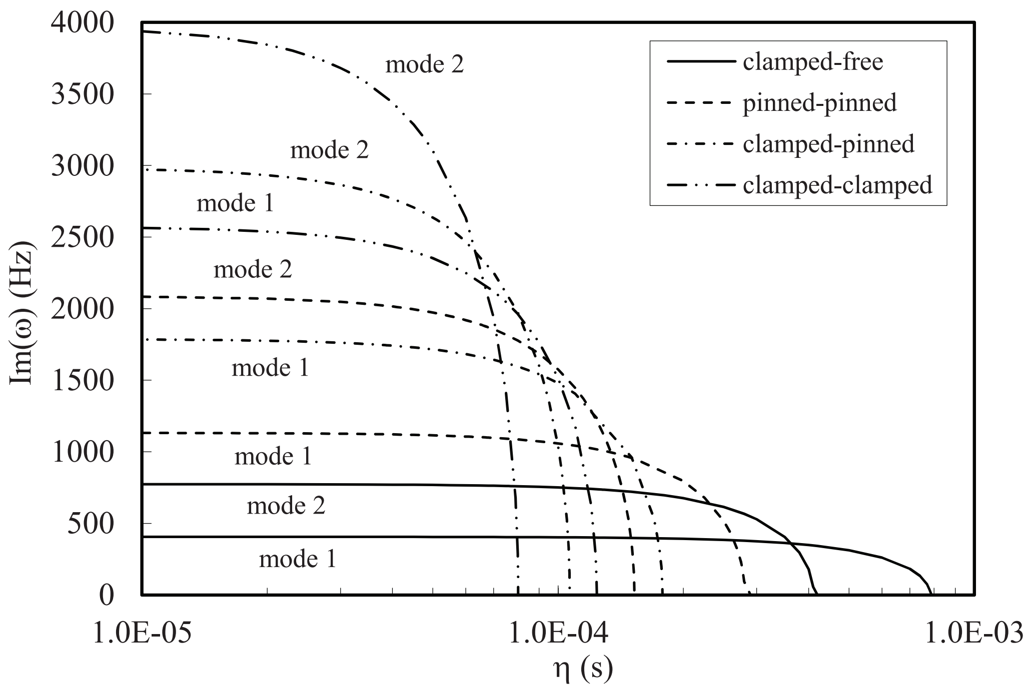


(b)

Fig. 2 Effect of internal damping and twist angle on eigenfrequencies for a damped clamped-free twisted beam: (a) damping part and (b) oscillating part.



(a)



(b)

Fig. 3 Effect of restraint types on eigenfrequencies for a damped twisted beam of $\phi = 45^o$: (a) damping part and (b) oscillating part.

$$(\mathbf{A}-\omega\mathbf{B})\mathbf{r}=0 \tag{11}$$

where

$$\mathbf{A}=\begin{bmatrix}\mathbf{O} & -\mathbf{K}\\ \mathbf{M} & \mathbf{O}\end{bmatrix} \quad \text{and} \quad \mathbf{B}=\begin{bmatrix}\mathbf{M} & \mathbf{C}\\ \mathbf{O} & \mathbf{M}\end{bmatrix}$$

The condition for the existence of the non-trivial solution of Eq, (11) is that the determinant of the coefficient matrix is zero.
Thus, the damped natural frequency can be obtained as the solution

$$\det(\mathbf{A}-\omega\mathbf{B})=0 \tag{12}$$

Generally, the eigenfrequency ω is a real number for the undamped beam but a complex number for the damped one. Therefore, the damped natural frequency has a real part and an imaginary part, which is related to the damping and oscillation, respectively. When the eigenfrequency ω has a negative real part, the system response will decay exponentially toward zero with time and the system is in stable condition.

4. Results and Discussions

The effect of the twist angle, internal damping and restraint types on the damped natural frequencies is demonstrated next. The material and geometric properties used in the calculations are E = 200 Gpa, ν= 0.3, ρ = 7860 kg m^{-3}, L = 0.2 m, b = 0.04 m, h = 0.02 m and κ = 5/6. The proportionality constants of the internal damping in bending and shearing, η_b and η_s, are taken to be the same and equal to η. The real and imaginary part of ω is denoted as Re(ω) and Im(ω).

Figure 2 shows the effects of the twisted angle and internal Kelvin-Voigt damping on the first two damped natural frequencies for clamped-free twisted beams. The absolute real (damping) parts are increased but imaginary (oscillating) parts are decreased with the increase in internal damping regardless of the twist angle. The oscillating parts reduce to zero values when the internal damping increases to a certain value. In addition, the oscillating part of the eigenfrequency of higher mode diminishes to zero at a lower critical internal damping. As can be seen, the absolute real part of the eigenfrequency of the higher mode is always greater than that of lower one. Thus, the higher vibration mode will be damped out more quickly. Depending on the value of damping, the influence of the twisted angle on the eigenfrequencies varies. For the twisted beam with damping less than a typical value, the oscillating part of eigenfrequency of mode 1 increases and that of mode 2 decreases with the increasing twist angle. However, for the beam with higher values of damping, the increase in the twist angle shows an opposite effect on the oscillating part of the eigenfrequency of mode 1 and 2. Regardless of the value of damping, the increasing twist angle increases the damping part of eigenfrequency of mode 1 and decreases that of mode 2. With the increase in twist angle, the critical internal damping at which the oscillating part of the eigenfrequency drops to zero decreases for mode 1 but increases for mode 2. Similar behaviours can also be observed for twisted beam with pinned-pinned, clamped-pinned and clamped-clamped boundary conditions.

Figure 3 presents the effect of various restraint types on the eigenfrequencies of mode 1 and 2 for twisted beams of twist angle $\phi = 45^{\circ}$ with various internal damping. The increase of restraint at the supporting ends always increases the absolute real part for each mode. However, the increasing end restraint increases the imaginary part only for the beam with Kelvin-Voigt damping lower than a typical value. When internal damping is greater than a typical value, the oscillating part of the beam with higher restraint might be smaller than that with less restraint depending on the internal damping value. As can be seen, the oscillating part of the higher restrained beam will diminish to zero at a lower internal damping for each mode. Since the damping part is much higher for the beam with higher restraint, this makes its solution highly damped.

5. Conclusion

Based on the results discussed earlier, some conclusions can be made as follows. The increasing Kelvin-Voigt damping increases the damping part and decreases the oscillating part of eigenfrequencies regardless of the twist angle. The increasing twist angle always increases the damping part of mode 1 and decreases that of mode 2. Depending on the value of Kelvin-Voigt damping, the increase in twist angle might increase or decrease the oscillating part of eigenfrequency. Generally, with the increase of restraint at the boundaries, both the damping and oscillating parts are increased as the Kelvin-Voigt damping is lower than a typical value.

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